

NOTES AND CORRESPONDENCE

Comments on “Improvement of the Short-Fetch Behavior in the Wave Ocean Model (WAM)”

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1. Introduction

In a recent paper, Hersbach and Janssen (1999) present a modification to the method used for integrating the source term in the third-generation ocean wave model (WAM). The new method shows substantial practical benefits when compared to the earlier method, but is still not entirely satisfactory for use in high-resolution coastal applications. More significantly, their work is based upon misunderstandings and errors, which they appear to have picked up from earlier publications. This note aims to take this opportunity to correct these long-established errors and clarify the situation. We demonstrate that the time-centered discretization used to integrate the source terms does not guarantee numerical stability, despite repeated claims in the literature to that effect. We also analyze and explain why the supposedly less accurate first-order implicit Euler method is generally superior to the second-order time-centered technique in most applications.

2. Background

WAM (Komen et al. 1994; WAMDI 1988) is a third-generation wave model designed for modeling the ocean wave field in deep water and up to global scales. In this context, third generation means that a full wave spectrum (F), discretized according to frequency and direction, is calculated diagnostically (up to a high-frequency threshold above which a prognostic tail is added) by finite difference approximations to partial differential equations.

WAM uses an operator splitting method in which the propagation terms and source terms are integrated sep-

arately. With the propagation terms excluded, the equation describing the evolution of F becomes

$$\frac{\partial F}{\partial t} = S(F), \quad (1)$$

where the $S(F)$ is the total source term. This is now a set of ordinary differential equations (ODEs) for which there are a wide range of well-researched solution methods (e.g., Press et al. 1994, chapter 16). Since the time-scale for the evolution of the high-frequency end of the spectrum is much shorter than for the low end of the spectrum (WAMDI 1988), this set of ODEs is in fact a *stiff* set of equations (Press et al. 1994, section 16.6). The literature on this particular subject (e.g., Seinfeld et al. 1970) does not seem to have been considered in the context of wave modeling, which may explain some of the errors and misunderstandings, which we address in the following sections.

3. Comments on the integration scheme

A simple method for the solution of Eq. (1) is to use the following finite difference approximation:

$$\frac{F_{n+1} - F_n}{\Delta t} = (1 - \alpha)S(F_n) + \alpha S(F_{n+1}). \quad (2)$$

Here F_n is the initial energy level, $F_{n+1} = F_n + \Delta F$ is the energy level after a finite time interval Δt has passed, and α is the implicitness parameter, which can be chosen in the range $[0, 1]$, with $\alpha = 0, 1/2, 1$ corresponding to the explicit (forward time) Euler method, time-centered and implicit (backward time) Euler method, respectively. We prefer to avoid using the term “semi-implicit” here since it has been used to mean different things by different authors. Also, $S(F_{n+1})$ is not known a priori but is estimated via a first-order Taylor series expansion around $S(F_n)$, which results in the following equation:

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$$\Delta F = \frac{\Delta t S(F_n)}{1 - \alpha \Delta t \min[S'(F_n), 0]}, \quad (3)$$

where $S'(F_n) = (\partial S / \partial F)|_{F_n}$.

The minimum operator is required to avoid the singularity and nonphysical results that could otherwise be produced if $S'(F_n)$ is positive. It is worth pointing out that when $S'(F_n)$ is positive, this equation reduces to the explicit Euler method, which is well known to exhibit poor stability (Press et al. 1994, chapter 16). Although the partial derivative of S is a matrix (when the spectrum is discretized), only the diagonal terms are retained since this greatly reduces the computation without significantly affecting the results (Komen et al. 1994, p. 235). This means that each equation can be solved independently without the need for matrix inversion.

The original integration scheme used in WAM was the time-centered method, $\alpha = 1/2$, with a time step that roughly matched the timescale of the evolution of low-frequency waves. It was stated by WAMDI (1988) and Komen et al. (1994) that this method was in principle numerically stable. Hersbach and Janssen (1999) repeat this assertion, referring to a "proof" by Janssen and Doyle (1997), and therefore they have to attribute the ensuing instability to the fact that the nondiagonal terms in the integration have been omitted. In fact, as has been stated (e.g., chapter 16.6 of Press et al. 1994 and chapter 111-F-5 of Roache 1976), the integration technique is *not* numerically stable, and counterexamples are readily found. For example, consider the ODE

$$\frac{dx}{dt} = -x^{1/3}. \quad (4)$$

It is trivial to solve this equation analytically, but the physically stable point at $x = 0$ will never be found by any of the three finite difference methods described above, for any finite time step. For a small value x_n close to the stable point, the implicit Euler method will calculate $x_{n+1} \approx -2x_n$ and so the distance from the equilibrium solution is amplified (the time-centered and explicit methods diverge even more rapidly). This example serves to disprove the claims cited above that the time-centered method is numerically stable.

This simple counterexample has a pathological singularity in its gradient at $x = 0$, but a slightly more contrived example can be created by replacing a short segment of the curve with a straight line segment; for example,

$$\frac{dx}{dt} = \begin{cases} -x^{1/3}, & |x| > 0.000001 \\ -10\,000x, & |x| \leq 0.000001. \end{cases} \quad (5)$$

The skeptical reader is invited to check that the integration of this ODE by any of the simple methods described above will not in general converge to the stable equilibrium point $x = 0$, although for this second example convergence is at least sometimes possible if one manages to reach the straight line segment either

through the use of an extremely short time step or fortuitous starting position. We emphasize that this second counterexample is extremely well behaved in the neighborhood of its stable fixed point (being linear around that point), but this does not help matters as the linear region will not necessarily be reached during a numerical integration.

Given that none of these three simple integration methods can guarantee unconditional stability, the preferred approach must surely be to reduce the time step where necessary. Indeed, Tolman (1992) has done this in his wave model WAVE-WATCH with apparently good results. Of course, the existence of pathological examples such as (4), which would require vanishingly small time steps, implies that no absolute guarantees of performance can be given.

The misunderstanding over the numerical stability of the schemes is compounded by the original choice of the time-centered scheme for the integration of the source terms. WAMDI (1988) asserts that "for high frequencies the method yields the (slowly changing) quasi-equilibrium spectrum" but in practice the method can generate oscillations in this region of the spectrum. Hersbach and Janssen (1999) observed such oscillations, and Janssen and Doyle (1997) attributed similar oscillations in the ECMWF atmospheric model to a spurious chaotic period-doubling effect. Both sets of authors discovered that the oscillations could be eliminated by the use of the implicit Euler method [$\alpha = 1$ in Eq. (3)]. The original time-centered scheme of WAM is not a generally favoured method for the integration of ODEs, and it is a particularly poor choice in the case of stiff sets of equations, where the implicit Euler method is undoubtedly a better choice (if more complex methods are prohibited by computational limitations). The time-centered technique was not even tested by Seinfeld et al. (1970) in their otherwise thorough review of simple methods for stiff ODEs, and this omission suggests that the drawback of this technique is well known in some quarters. Since this does not, however, appear to be widely known in the oceanographic and meteorological modeling communities, these comments (which apply widely across many numerical models and not just to wave modeling) are fully justified in the appendix.

4. Numerical versus physical limitation

The approach used in WAM to cope with the numerical instability of the integration method is to apply a "limiter," which restricts the rate of change of the spectrum. Hersbach and Janssen (1999) note that their limiter acts as if it is part of the physics of the model since its influence remains even as $\Delta t \rightarrow 0$. They assert that this alteration to the physical description of wave evolution is justified since it "compensates for the lack of physics in the diagnostic part of the spectrum." There is no description or justification of this new physics in

WAMDI (1988) or Komen et al. (1994), or even of the limiter itself, although it was apparently in use at that time, and indeed the only previous peer-reviewed publication that we can find that mentions the existence of the limiter appears to be Tolman (1992). Furthermore, Hersbach and Janssen (1999) show in their own work that any modification to the physics is not justified. When they run WAM for a small enough time step ($\Delta t = 1$ s), the limiter is not needed and the output appears to be physically satisfactory. Indeed, Hersbach and Janssen (1999) judge the performance of their new proposed limiter in part by comparing its results to the convergent no-limiter WAM output. They argue that the new limiter is superior to the old one because its physical impact is very much lower, which appears to be in direct contradiction to the earlier statement that a change to the model physics is required. The convergent no-limiter runs demonstrate that the limiter is not required to compensate for the lack of physics in the diagnostic model. Instead, as we have explained above, it corrects for a numerically unstable integration scheme. In practical terms, the new limiter does a fairly good job, especially over larger grids, but there is still a noticeable effect in coastal regions, which leads to an underprediction of wave height. Monbaliu et al. (2000) explore this issue further and present the results from an alternative method of source term integration, which does converge to the continuum solution as the time step is reduced. For the reasons described above, however, this method (which does not rely on a limiter) cannot be guaranteed to always be stable, and so is not suitable for operational use.

It is of course always possible that the physical description of wave evolution could be improved in fetch-limited situations, but no evidence for this is supplied. The imposition of a limiter can only make any improvement to the physical equations more difficult since it means that a change to (for example) the source terms might not fully feed through into the model output due to the model run being constrained by the limiter rather than solving the underlying physical equations. It must be recognized that the need for a limiter is a purely numerical artifact, made necessary by the use of an unstable integration method together with a large time step. The numerical instability is not related to the physics of wave growth (as the $\Delta t = 1$ s model runs prove), and the post hoc rationalization by Hersbach and Janssen (1999) of the limiter as physically based is completely unjustified since it in truth compensates for a numerically unstable integration method.

5. Conclusions

None of the integration methods discussed by Hersbach and Janssen (1999) can guarantee stability for an arbitrary time step, despite several claims in the literature to this effect, and the limiter is required for purely numerical reasons. Where a relatively large time step is

used (as will often be the case when, as here, the set of ODEs is stiff), the time-centered discretization should be avoided since it generates nonphysical oscillatory behavior and converges only slowly to a quasi-equilibrium solution. Forward-time integration is preferable in both theory and practice, and reduces the likelihood of oscillations, although there are no simple guarantees of stability. A mathematically and physically better solution to this problem of instability would be to reduce the time step where the spectrum is evolving rapidly, and this method has been employed by Tolman (1992). However, for most applications, the method presented by Hersbach and Janssen (1999) seems to work well enough (the problems are restricted to high-resolution coastal applications) despite its dubious theoretical foundations.

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APPENDIX

Analysis of Integration Methods

Error analysis for finite difference methods is generally performed by comparing the difference equations to a Taylor series expansion of the continuum equation. For example, a second-order Taylor series expansion of F_{n+1} around F_n using (1) produces

$$F_{n+1} = F_n + \Delta t S(F_n) + \frac{(\Delta t)^2}{2} S(F_n) S'(F_n), \quad (\text{A1})$$

which can be rearranged as

$$\frac{\Delta F}{\Delta t} = S(F_n) \left[1 + \frac{\Delta t}{2} S'(F_n) \right]. \quad (\text{A2})$$

A first-order Maclaurin expansion of the fraction in eq. (3) yields

$$\frac{\Delta F}{\Delta t} = S(F_n) [1 + \alpha \Delta t S'(F_n)]. \quad (\text{A3})$$

Inspection of (A3) and (A2) indicates that a second-order accurate (in Δt) finite difference equation is generated by choosing the value $\alpha = 1/2$. This value corresponds to the time-centered discretization.

However, for large Δt , this analysis is not relevant or even valid. First, the higher-order terms in the Taylor series expansion (A1) may be larger than the low-order terms that have been retained, and more importantly, the Maclaurin expansion in (A3) is only convergent for $|\alpha \Delta t S'(F_n)| < 1$. An alternative approach reveals the truth of the situation.

We can sidestep the problem of a large Δt by using a first-order Taylor series expansion for $S(F)$ around $S(F_n)$ and solving equation (1) directly. The truncated Taylor series is given by

$$S(F) = S(F_n) + (F - F_n)S'(F_n). \quad (\text{A4})$$

The error of this approximation depends on $F - F_n$, which (for physically stable equations) we can reasonably expect to be finite and bounded even for an arbitrary time step. Substituting this expression into (1) enables the resulting linear ODE to be integrated analytically, and the solution for F_{n+1} is given by

$$F_{n+1} = F_n - \frac{S(F_n)}{S'(F_n)}[1 - e^{\Delta t S'(F_n)}]. \quad (\text{A5})$$

Of course, the accuracy with which this matches the exact solution of the original ODE depends on how well the first-order Taylor series approximates the source function (i.e., it is exact in the linear case, and the error depends on the degree of nonlinearity in the source function). If $S'(F_n)$ is negative (as we would expect in the region of a physically stable root), then the exponential term in (A5) vanishes for large Δt and the resulting equation converges to

$$\Delta F = \frac{-S(F_n)}{S'(F_n)}. \quad (\text{A6})$$

This equation will be recognized as the Newton–Raphson iteration scheme for finding a root of the equation $S(F) = 0$ (Press et al. 1994, chapter 9.4). This is encouraging since the physically correct solution for the ODE is that F should converge to an equilibrium point over time, and any equilibrium point is of course a root of the equation $S(F) = 0$. It will now be shown that the implicit Euler scheme approaches this solution as Δt increases but the time-centered scheme does not.

Assuming for clarity that the derivative S' is non-positive (this is required in the neighborhood of a stable equilibrium point) and taking $\alpha = 1$, we can rearrange Eq. (3) to give

$$\Delta F = \frac{-S(F_n)}{S'(F_n) - 1/\Delta t}. \quad (\text{A7})$$

The only difference between Eqs. (12) and (11) is the $1/\Delta t$ term in the denominator. For a large time step this extra term is vanishingly small, so the implicit Euler method generally works reasonably well even with a large time step.

If, however, we consider the time-centered method, $\alpha = 1/2$, Eq. (3) can be rearranged to give

$$\Delta F = \frac{-2S(F_n)}{S'(F_n) - 2/\Delta t}. \quad (\text{A8})$$

The method is still similar to Newton–Raphson, but the factor of 2 in the numerator effectively acts as an over-relaxation parameter, which if applied to the Newton–Raphson method directly, would completely prohibit convergence (Press et al. 1994, chapter 19.5). In fact, a slow convergence will generally occur due to the extra $2/\Delta t$ term in the denominator, and the form of this term means that the convergence will deteriorate as the time

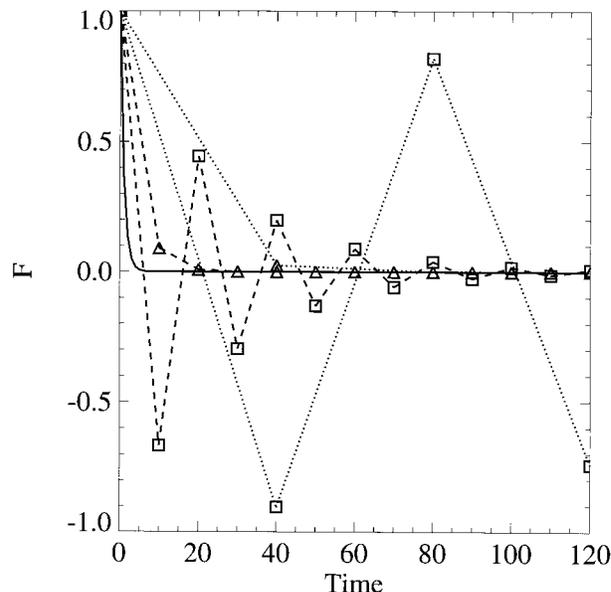


FIG. A1. Finite difference solutions to $\partial F/\partial t = -F$. Solid line is exact solution, triangles show implicit Euler method, and squares show time-centered method. Dotted line is for $\Delta t = 40$, dashed line for $\Delta t = 10$.

step increases. Figure A1 shows a comparison between the solutions generated by various implementations of the methods described. The underlying differential equation is given by

$$\frac{\partial F}{\partial t} = -F, \quad (\text{A9})$$

where for simplicity, we are considering F as a single variable and the initial condition is taken to be $F(0) = 1$. The implicit Euler method gives reasonable results for both time steps used, but the time-centered technique generates oscillations which are substantially worse for the larger time step, as explained above.

When a very small time step is used, the time-centered method may be marginally more accurate than the first-order explicit and implicit Euler methods, but in this case they will all be close to the continuum solution and so errors from other terms in a system of PDEs (advection, in the case of WAM) will probably render the differences insignificant. For a large time step, the time-centered method generates nonphysical oscillations even for a simple linear ODE and the implicit Euler method is to be preferred. For these reasons, it seems that the time-centered technique is a poor choice that is best avoided for integration of the source terms in a system of PDEs, unless there are very specific requirements for it.

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